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Turbulence generated by the interaction of entropy fluctuations with non-uniform mean flows

By M. E. GOLDSTEIN

National Aeronautics and Space Administration, Lewis Research Center, Cleveland, Ohio 44135

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This paper is primarily concerned with the turbulence generated by the interaction of random entropy fluctuations with non-uniform mean flows. The results are obtained by using rapid distortion theory in conjunction with a high frequency solution of a previously developed wave equation that governs the small amplitude unsteady vortical and entropic motion on steady potential flows (Goldstein 1978). The analysis is applied to symmetric contractions or expansions (or combinations of the two) and comparisons are made with a corresponding theory of Batchelor & Proudman (1954) for the rapid distortion of upstream turbulence by a contraction of this type. It is shown that the energy of the entropy-generated turbulence increases much more rapidly with contraction ratio than that of the imposed upstream turbulence. Turbulence of the former type can therefore dominate the latter even when the upstream entropy fluctuations are weak relative to the imposed turbulent intensities. Moreover the structure of the entropy-generated turbulence is found to be quite different from that of the contracted upstream turbulence.

A secondary purpose of the paper is to discuss some effects of compressibility on rapid distortion theory.

1. Introduction

There have been a large number of studies of the sound fields produced by entropy fluctuations passing through nozzles and other regions of non-uniform flow (Candel 1972; Cumpsty & Marble 1974; Pickett 1974; Ffowcs Williams & Howe 1975). This work has, to a large extent, been motivated by the need to understand the combustion or core noise generated by entropy fluctuations in jet engine combustors. The interaction of these entropy fluctuations with the non-uniform downstream mean flow can also produce turbulence and other unsteady flows that can effect boundary-layer separation and transition and can contribute significantly to the unsteady turbine blade loads that result in reduced fatigue life. Since there is considerable interest in improving the fatigue life of gas turbine blades, it is important to study this entropy-generated turbulence. Needless to say, such studies are also important in their own right.

In this paper we consider the turbulence that is generated when random entropy fluctuations are imposed on steady potential flows that would otherwise have uniform temperature and velocity at upstream infinity. As they are convected downstream the random upstream entropy fluctuations interact with the mean flow to produce

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velocity fluctuations that are also random functions of space and time. Such velocity fluctuations must certainly be thought of as turbulence.

We shall suppose that the ratio β of the temperature fluctuations, associated with the upstream entropy fluctuations, to the absolute mean flow temperature is small and that the mean flow and turbulence Reynolds numbers are large. In fact we shall require that β be small enough to ensure that $\beta(a/l) \ll 1$ even when the scale l of the entropy fluctuations is small compared to the characteristic dimension a of the mean flow velocity gradients. Then a trivial modification of the argument used by Hunt (1973) for the distortion of imposed turbulence will show that turbulent motion can be treated as a small inviscid perturbation about a steady potential flow.

A general theory for the small amplitude unsteady vortical and entropic motion on a potential flow was developed by Goldstein (1978), hereafter referred to as I. It was shown that the unsteady flow can be calculated by solving a single second-order linear wave equation. There is no essential restriction on the Mach number at which this result applies but, for definitiveness, its derivation was carried out for external flows with no incident acoustic fields. However there are many internal flows, especially in engineering devices where there is a premium in weight and space, that have uniform upstream conditions and undergo large accelerations. The frictional effects of the walls cannot significantly penetrate into the main stream of such flows and they will consequently contain large regions of nearly potential flow. In § 2 we show that the results of I apply, with only trivial modification, to flows of this type. They can therefore be used to calculate the entropy-generated turbulence described above even for internal flows.

This linearized calculation of the turbulence represents an extension of the rapid distortion theory developed by Batchelor & Proudman (1954) and Ribner & Tucker (1953), and generalized by Hunt (1973). In order to obtain simple formulas we consider only the classical rapid distortion limit where $a/l \ll 1$.

In § 3 we obtain the high frequency solution to the linear wave equation given of I. The result applies for arbitrary upstream distortions of the mean potential flow that can in general consist of both entropy and velocity fluctuations. We then show that the portion of the solution associated with the upstream velocity fluctuations can be transformed into the one used by Batchelor & Proudman (1954) for an incompressible flow and that it generalizes the result used by Ribner & Tucker (1953) for a compressible flow. This part of the solution is therefore not considered further.

But in § 4 the portion of the solution associated with the entropy fluctuations is used to relate the turbulent velocity fluctuations at any point in the flow to the spectrum of a random upstream entropy field, which is assumed to be homogeneous.

Simple results are obtained for the case of a symmetric contraction, expansion or combination of the two. In § 5 their physical implications are discussed for the case where the ratio c of the downstream to upstream velocities is greater than unity. The parameter c can be interpreted as the contraction ratio of the stream when the mean flow Mach number is not too large. It is shown that the energy of the entropy-generated turbulence eventually increases as c^2 for large values of c while classical rapid distortion theory shows that the energy of the upstream imposed turbulence increases only linearly with c. Then even when the ratio β of the upstream temperature fluctuations to the mean temperature is small compared to the ratio of the upstream velocity fluctuations to the mean velocity, the entropy-generated turbulence can eventually exceed the imposed turbulence – which is important to keep in mind when estimating the unsteady loads on obstacles placed in the stream.

Only the energy in the axial velocity components of the entropy-generated turbulence increases with c; the energy of the transverse velocity components decreases like $(1/c)\ln c$ for large c. This is again distinctly different from the behaviour of the imposed turbulence where transverse velocity energy increases with c and axial velocity energy decreases.

For large values of c the one-dimensional spectrum of the turbulent energy becomes equal to the one-dimensional spectrum of the entropy fluctuations that produced the turbulence.

2. The governing equations

The most general small amplitude non-acoustic upstream distortion that can be imposed on a steady potential flow with otherwise uniform upstream conditions consists of a frozen solenoidal velocity field

$$\mathbf{u}_\infty(x_1 - U_\infty t, x_2, x_3), \quad \nabla . \mathbf{u}_\infty = 0; \tag{2.1}$$

a frozen entropy field

$$s_\infty(x_1 - U_\infty t, x_2, x_3); \tag{2.2}$$

and *no* unsteady pressure field. Here U_∞ denotes the constant mean flow velocity at upstream infinity, t denotes the time, x_1, x_2, x_3 are Cartesian co-ordinates with x_1 in the upstream mean flow direction. The quantities $\mathbf{u}_\infty$ and s_∞ can be taken to be any functions of their arguments. It was shown in I that the resulting perturbation in velocity

$$\mathbf{u} = \nabla\phi + \mathbf{u}^{(I)}, \tag{2.3}$$

at any point of the flow will then consist of a rotational part $\mathbf{u}^{(I)}$ that is independent of the pressure fluctuations and is a known function of the upstream distortions $\mathbf{u}_\infty$ and s_∞ and the mean flow quantities; and an irrotational part $\nabla\phi$ that is related to the pressure fluctuations p' by

$$p'/\rho_0 = -D_0\phi/Dt, \tag{2.4}$$

where $D_0/Dt = \partial/\partial t + \mathbf{U}.\nabla$ is the convective derivative based on the mean flow velocity $\mathbf{U} = \{U_1, U_2, U_3\}$. $\rho_0 = \rho_0(\mathbf{x})$ is the mean flow density and $\mathbf{x} = \{x_1, x_2, x_3\}$ is the position vector.

The known contribution $\mathbf{u}^{(I)} = \{u_1^{(I)}, u_2^{(I)}, u_3^{(I)}\}$ is given by

$$u_i^{(I)} = \mathbf{u}_\infty(\mathbf{X} - \hat{\mathbf{i}}U_\infty t) . \frac{\partial \mathbf{X}}{\partial x_i} + \frac{1}{2c_p} s_\infty(\mathbf{X} - \hat{\mathbf{i}}U_\infty t)\frac{\partial}{\partial x_i}(\Phi - U_\infty X_1) \quad \text{for} \quad i = 1, 2, 3, \tag{2.5}$$

where $\hat{\mathbf{i}}$ is a unit vector in the x_1 direction, Φ is the mean flow velocity potential, c_p is its specific heat at constant pressure (which is assumed to be constant) and

$$\mathbf{X} = \{X_1, X_2, X_3\}$$

is a vector function of $\mathbf{x}$ such that $\mathbf{X} \to \mathbf{x}$ as $x_1 \to -\infty$. The intersections of the surfaces $X_2 = \text{constant}$ and $X_3 = \text{constant}$ define the mean flow streamlines as shown

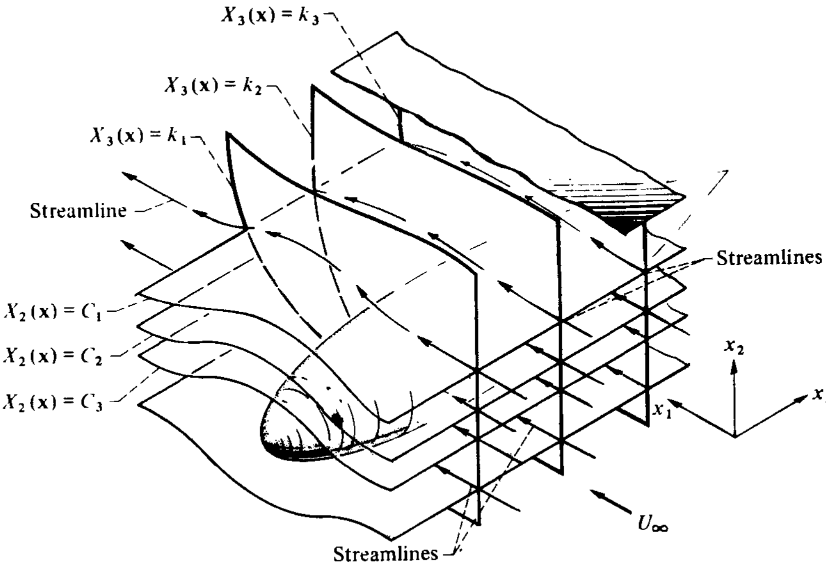

FIGURE 1. Illustration of X_2 and X_3 surfaces.

schematically in figure 1. The quantity X_1/U_∞ is the Lighthill (1956)–Darwin (1953) 'drift' function

$$\frac{x_1}{U_\infty} + \int_{-\infty}^{x_1} \left[\frac{1}{U_1(x_1', y_s(x_1', X_2, X_3), z_s(x_1', X_2, X_3))} - \frac{1}{U_\infty}\right] dx_1',$$

where x_1', $y_s(x_1', X_2, X_3)$, $z_s(x_1', X_2, X_3)$ are the points of the streamline defined by the intersections of the surfaces X_2 = constant, X_3 = constant that pass through the point x_1, x_2, x_3. The change in X_1/U_∞ between any two points on this streamline is equal to the time it takes a mean flow fluid particle to traverse the distance between those points. The quantity $X_1 - U_\infty t$ will therefore remain constant relative to an observer moving downstream with the mean flow and since X_2 and X_3 are constant along all mean flow streamlines, they will also have this property.

Then since D_0/Dt denotes differentiation following a fluid particle along its streamline and since $\mathbf{X} - \hat{\mathbf{i}} U_\infty t = \{X_1 - U_\infty t, X_2, X_3\}$, it follows that

$$\frac{D_0}{Dt}(\mathbf{X} - \hat{\mathbf{i}} U_\infty t) = 0. \tag{2.6}$$

The components of the vector $\mathbf{X} - \hat{\mathbf{i}} U_\infty t$ are essentially Lagrangian co-ordinates of the mean flow fluid particles.

Finally, the 'perturbation potential' ϕ (which is the only perturbation quantity that is not known) can be found by solving the linear inhomogeneous wave equation

$$\frac{D_0}{Dt}\frac{1}{c_0^2}\frac{D_0\phi}{Dt} - \frac{1}{\rho_0}\nabla.(\rho_0\nabla\phi) = \frac{1}{\rho_0}\nabla.\rho_0\mathbf{u}^{(I)}, \tag{2.7}$$

where $c_0 = c_0(\mathbf{x})$ is the mean flow sound speed. This equation has a dipole-type source term where strength $\rho_0 \mathbf{u}^{(I)}$ is completely known in terms of the mean flow quantities and the upstream boundary conditions $\mathbf{u}_\infty$ and s_∞. Once its solution has been found all other flow quantities can be calculated by differentiation.

When rigid surfaces are present, equation (2.7) must be solved subject to the boundary condition

$$\hat{\mathbf{n}}.\nabla\phi \to -\hat{\mathbf{n}}.\mathbf{u}^{(I)} \quad \text{as} \quad \mathbf{x} \to \text{boundary}, \tag{2.8}$$

where $\hat{\mathbf{n}}$ is the unit normal to these surfaces.

It is known that nonlinear effects can be important when small amplitude waves propagate upstream in a region of flow that is decelerating through Mach number 1 (see Landahl 1961, pp. 12–14; Myers & Callegari 1978; Prasad & Krishnan 1977). We shall, therefore, suppose that the mean flow is always nondecelerating in any region where the local Mach number is close to unity. Then the present results should apply at any Mach number when the flow is internal to the bounding surfaces. But, as indicated in the introduction, the derivation of I was, for definiteness, limited to external flows with no incident acoustic fields. We shall now show that the results apply, with only trivial modification, to any inviscid small amplitude motion on any steady potential flow (external or internal) that has uniform upstream conditions and is bounded by rigid surfaces.

Since these flows are all described by the same linearized equations and involve the boundary condition (2.8), equations (2.3) and (2.4) with $\mathbf{u}^{(I)}$ given by (2.5) and ϕ determined by (2.7) and (2.8) will certainly provide a solution to the governing equations of any of these flows and this solution can always be made to satisfy appropriate boundary conditions on the rigid surfaces. It is therefore only necessary to show that it can also be made to satisfy appropriate conditions at upstream infinity where the mean flow is uniform.

Now it was shown by Kovàsznay (1953) that the most general small amplitude motion on a uniform flow has a velocity field, say $\boldsymbol{u}^{\dagger}$, that can be decomposed into the sum

$$\boldsymbol{u}^{\dagger} = \nabla\phi_{\infty} + \boldsymbol{u}_{\infty}, \tag{2.9}$$

of (i) a frozen solenoidal velocity field $\boldsymbol{u}_{\infty}$ that is given by (2.1) and is completely decoupled from the pressure and entropy fluctuations and (ii) an irrotational disturbance $\nabla\phi_{\infty}$ that produces no entropy fluctuations but is directly related to the pressure fluctuations and is as a result connected with any acoustic motion that may occur. Finally the fluctuations in entropy are given by (2.2) and are decoupled from the velocity and pressure fluctuations (but produce density fluctuations). The potential ϕ_{∞} satisfies a linear homogeneous wave equation with constant coefficients. A simple but rigorous proof of these assertions can be found in Goldstein (1976), pp. 220–221.

For the present purpose the important thing to notice is that each of these modes of motion (i.e. the vorticity mode, the pressure mode and the entropy mode) is itself a solution of the governing equations. In fact (2.1) and (2.2) will satisfy these equations for any functions $\mathbf{u}_{\infty}$ and s_{∞} that have the indicated arguments and, since they represent disturbances that are convected downstream, these functions can still be specified as upstream boundary conditions. The pressure mode $\nabla\phi_{\infty}$ will on the other hand correspond to actual acoustic motion since we are dealing with the region at upstream infinity. The downstream propagating acoustic waves can be imposed as upstream boundary conditions but the upstream propagating waves must be determined by the solution. We can, for example, ensure that there will be only outward propagating waves by imposing a Summerfeld radiation condition.

For the external flows considered in I all outgoing waves decayed and the incident acoustic fields were eliminated by imposing the condition $\phi \to 0$ as $x_1 \to -\infty$. Now even when this boundary condition is not imposed, (2.3) through (2.7) still provide a solution to the appropriate governing equations and $\mathbf{u}^{(I)}$ and the entropy fluctuations still approach (2.1) and (2.2) as $x_1 \to -\infty$. Then since ρ_0 and $\mathbf{U}$ become constant as $x_1 \to -\infty$, (2.7) will reduce to the homogeneous acoustic wave equation with constant coefficients that governs the upstream pressure mode ϕ_∞. Hence it follows from (2.3) that the perturbation velocity $\mathbf{u}$ will approach the form (2.9) and that ϕ can be identified with ϕ_∞ as $x_1 \to -\infty$. The incident acoustic waves can be eliminated by requiring that ϕ satisfy a radiation condition rather than the stronger condition $\phi \to 0$ as $x_1 \to -\infty$ that was imposed in I or we can, more generally, specify the portion of ϕ that does not satisfy this radiation condition. The upstream vortical velocity and entropy fluctuations (2.1) and (2.2) can still, of course, be independently specified as upstream boundary conditions. Then with this slight modification of the upstream conditions the results of I will apply to the unsteady motion on *any* steady potential flow with uniform upstream conditions provided it does not decelerate through Mach number 1.

3. High frequency solution to the wave equation for small distortions of a potential flow

We are interested here in flows that exist for all time. Then, as explained in I, we need only consider an incident harmonic disturbance field of the type

$$\left.\begin{aligned} \mathbf{u}_\infty &= \mathbf{A}\exp[i\mathbf{k}.(\mathbf{x}-\mathbf{i}U_\infty t)] \\ s_\infty &= B\exp[i\mathbf{k}.(\mathbf{x}-\mathbf{i}U_\infty t)] \end{aligned}\right\} \quad \text{as} \quad x_1 \to -\infty, \tag{3.1}$$

where $\mathbf{A}$, B, and $\mathbf{k}$ are, aside from the requirement that

$$\mathbf{A}.\mathbf{k} = 0, \tag{3.2}$$

arbitrary constants. The solution for any given incident distortion field can then be obtained by superposing solutions of this type, i.e. by letting $\mathbf{A}$ and B depend on $\mathbf{k}$ and integrating the solutions over $\mathbf{k}$.

Substituting (3.1) into (2.5) we find

$$\mathbf{u}^{(I)} = \exp[i\mathbf{k}.(\mathbf{X}-\mathbf{i}U_\infty t)]\,\nabla[\mathbf{A}.\mathbf{X}+(B/2c_p)(\Phi - U_\infty X_1)]. \tag{3.3}$$

If in (2.3), (2.7) and (3.3) we non-dimensionalize all lengths, including $\mathbf{X}$, by a characteristic spatial scale a of the mean flow velocity gradients, the time by a/U_∞, the mean flow velocity by U_∞ and ϕ by $|\mathbf{A}|a$ (or by aBU_∞/c_p), the wavenumber vector $\mathbf{k}$ will appear only in the combination $a\mathbf{k}$ and the vector $(\mathbf{X}-\mathbf{x})$ and its derivatives will be of order one (except perhaps near the boundaries where X_1 can become infinite).

Now suppose that

$$a|\mathbf{k}| \gg 1. \tag{3.4}$$

Then substituting (3.3) into the right side of (2.7) yields

$$\frac{1}{|\mathbf{k}|}\frac{1}{\rho_0}\nabla.\rho_0\mathbf{u}^{(I)} = \frac{i}{|\mathbf{k}|}\left\{\frac{\partial}{\partial x_j}\left[\mathbf{A}.\mathbf{X}+\frac{B}{2c_p}(\Phi - UX_1)\right]\right\}\left[\frac{\partial}{\partial x_j}(\mathbf{k}.\mathbf{X})\right]$$
$$\times\exp[i\mathbf{k}.(\mathbf{X}-\mathbf{i}U_\infty t)] + O((a|\mathbf{k}|)^{-1}) \quad \text{as} \quad a|\mathbf{k}| \to \infty. \tag{3.5}$$

Hence it is natural to seek a solution to (2.7) of the form

$$\phi = \frac{1}{|\mathbf{k}|} D(\mathbf{x}, t) \exp[i\mathbf{k}.(\mathbf{X} - \mathbf{\hat{\imath}} U_\infty t)], \tag{3.6}$$

where D is a 'slowly varying' function of $\mathbf{x}$ which together with its partial derivatives with respect to x_i/a is assumed to be of order 1. In fact, substituting (3.5) and (3.6) and using (2.6) yields

$$\frac{1}{|\mathbf{k}|^2} D|\nabla(\mathbf{k}.\mathbf{X})|^2 \exp[i\mathbf{k}.(\mathbf{X} - \mathbf{\hat{\imath}} U_\infty t)] = \frac{i}{|\mathbf{k}|}\left\{\frac{\partial}{\partial x_j}\left[\mathbf{A}.\mathbf{X} + \frac{B}{2c_p}(\Phi - U_\infty X_1)\right]\right\}$$
$$\times \left[\frac{\partial}{\partial x_j}(\mathbf{k}.\mathbf{X})\right] \exp[i\mathbf{k}.(\mathbf{X} - \mathbf{\hat{\imath}} U_\infty t)] + O((a|\mathbf{k}|)^{-1}) \quad \text{as} \quad a|\mathbf{k}| \to \infty.$$

Hence

$$D = \frac{i|\mathbf{k}|\boldsymbol{\chi}.\nabla\left[\mathbf{A}.\mathbf{X} + \dfrac{B}{2c_p}(\Phi - U_\infty X_1)\right]}{\chi^2} + O\left(\frac{1}{a|\mathbf{k}|}\right) \quad \text{as} \quad a|\mathbf{k}| \to \infty, \tag{3.7}$$

where we have put

$$\boldsymbol{\chi} = \{\chi_1, \chi_2, \chi_3) = \nabla(\mathbf{k}.\mathbf{X}) \tag{3.8}$$

and

$$\chi = |\boldsymbol{\chi}|. \tag{3.9}$$

Inserting (3.7) into (3.6) and using the result together with (3.3) in (2.3) we obtain

$$u_i = \left(M_{ij} A_j + \frac{1}{2c_p} N_i B\right) \exp[i\mathbf{k}.(\mathbf{X} - \mathbf{\hat{\imath}} U_\infty t)], \tag{3.10}$$

where

$$M_{ij} \equiv \frac{\chi^2 \partial X_j/\partial x_i - \chi_l \chi_i \partial X_j/\partial x_l}{\chi^2}, \tag{3.11a}$$

$$N_i \equiv \frac{\chi^2(\partial/\partial x_i)(\Phi - U_\infty X_1) - \chi_i \chi_l (\partial/\partial x_l)(\Phi - U_\infty X_1)}{\chi^2} \tag{3.12a}$$

and it is understood that these formulas are to be summed from one to three on the repeated indices.

These results can be written more compactly by introducing the permutation tensor ϵ_{ijk} (ϵ_{ijk} is equal to zero when any two indices are equal, equal to 1 when they are equal to a cyclic permutation of 1, 2, 3 and equal to -1 otherwise) to obtain

$$M_{ij} = \frac{\chi_n \chi_m}{\chi^2} \epsilon_{kni} \epsilon_{kml} \frac{\partial X_j}{\partial x_l}, \tag{3.11b}$$

$$N_i = \frac{\chi_n \chi_m}{\chi^2} \epsilon_{kni} \epsilon_{kml} \frac{\partial}{\partial x_l}(\Phi - U_\infty X_1), \tag{3.12b}$$

which shows that N_i is just the ith component of the vector triple product

$$-\boldsymbol{\chi} \times \boldsymbol{\chi} \times [\nabla(\Phi - U_\infty X_1)]/\chi^2.$$

Since $\mathbf{X} \to \mathbf{x}$ and $\boldsymbol{\chi} \to \mathbf{k}$ as $x_1 \to -\infty$, (3.2) and (3.10) imply that $\mathbf{u} \to \mathbf{u}_\infty$ as $x_1 \to -\infty$. Then the pressure mode $\nabla\phi$ vanishes as $x_1 \to -\infty$ and the upstream acoustic waves will be of higher order in $(a|\mathbf{k}|)^{-1}$ as $a|\mathbf{k}| \to \infty$. In fact, since (3.6) represents a disturbance which is locally frozen in the flow and which produces pressure fluctuations that are only $O((a|\mathbf{k}|)^{-1})$, the potential disturbance $\nabla\phi$ is definitely non-acoustic everywhere in the flow. The acoustic motion will therefore not enter the analysis to the order of this solution and need not be considered further. This is not unexpected since the high frequency vortical motion (which the present solution represents) behaves locally as if it were propagating through a uniform flow and, as we have seen in § 2, this produces no acoustic motion.

The solution (3.10) through (3.12) will, in general, not satisfy the solid surface boundary condition (2.8). But, as shown by Hunt (1973) for the incompressible case, this condition can only effect the solution within a region $O((a|\mathbf{k}|)^{-1})$ surrounding the boundary and can therefore be ignored.

The first term in (3.10) obviously represents the unsteady velocity field produced by the imposed upstream velocity perturbation $\mathbf{u}_\infty$. When the mean flow is incompressible, it can be transformed into the result used by Batchelor & Proudman (1954) for that case (see appendix) and for a compressible flow it generalizes the result obtained by Ribner & Tucker (1953) for the special case where $\partial X_i/\partial x_j$ is zero if $i \neq j$. We shall therefore not consider this term further. The second term in (3.10) represents the unsteady velocity field produced by the imposed upstream entropy fluctuations s_∞. It will be used to calculate the turbulence produced by a random entropy field. In order to do this it is first necessary to relate the turbulence velocity correlations at any point of the flow to appropriate correlation spectra of the upstream disturbance field.

4. Calculation of velocity correlations due to random homogeneous incident distortion

It is easy to see that the velocity field produced by an arbitrary incident harmonic distortion can be expressed in the form (3.10) even when $|\mathbf{k}|a$ is not assumed to be large. Thus we can always write

$$u_i = \left[\mathscr{M}_{ij}(\mathbf{x}\mid\mathbf{k})\,A_j + \frac{1}{2c_p}\mathscr{N}_i(\mathbf{x}\mid\mathbf{k})\,B\right]\exp(-ik_1U_\infty t), \quad i = 1,2,3, \tag{4.1}$$

where $\mathscr{M}_{ij}$ and $\mathscr{N}_i$ are independent of t, A_j and B. Then when $|\mathbf{k}|a$ is large [see (3.10)],

$$\mathscr{M}_{ij} = M_{ij}\,e^{i\mathbf{k}.\mathbf{X}} \quad \text{and} \quad \mathscr{N}_i = N_i\,e^{i\mathbf{k}.\mathbf{X}}. \tag{4.2}$$

When a statistically homogeneous but otherwise arbitrary random velocity fluctuation is imposed on the flow at upstream infinity, the velocity correlation at any other point of the flow

$$R_{ij}(\mathbf{x},\mathbf{x}',\tau) \equiv \overline{u_i(\mathbf{x},t)\,u_j(\mathbf{x}',t+\tau)} \tag{4.3}$$

is related to the spectrum

$$\Phi_{ij}^{(\infty)}(\mathbf{k}) \equiv \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} R_{ij}^{(\infty)}(\mathbf{y})\,e^{-i\mathbf{k}.\mathbf{y}}\,d\mathbf{y} \tag{4.4}$$

of the upstream velocity covariance tensor

$$R_{ij}^{(\infty)}(\mathbf{y}) = \overline{u_{\infty i}(\mathbf{x}-\hat{\mathbf{i}}U_\infty t)\,u_{\infty j}(\mathbf{x}+\mathbf{y}-\hat{\mathbf{i}}U_\infty t)} \tag{4.5}$$

(which, owing to the assumed homogeneity, depends only on the indicated argument) by (Hunt 1973, 1977)

$$R_{ij}(\mathbf{x},\mathbf{x}',\tau) \equiv \iiint_{-\infty}^{\infty} \mathscr{M}_{ik}^{*}(\mathbf{x}\mid\mathbf{k})\,\mathscr{M}_{jl}(\mathbf{x}'\mid\mathbf{k})\,\Phi_{kl}^{(\infty)}(\mathbf{k})\exp(-ik_1U_\infty\tau)\,d\mathbf{k}, \tag{4.6}$$

where the asterisk denotes the complex conjugate. Then the one-dimensional spectrum

$$\Theta_{ij}(\mathbf{x},\mathbf{x}' \mid k_1) \equiv \frac{U_\infty}{2\pi}\int_{-\infty}^{\infty} R_{ij}(\mathbf{x},\mathbf{x}',\tau)\exp(ik_1 U_\infty \tau)\,d\tau \tag{4.7}$$

is given by

$$\Theta_{ij}(\mathbf{x},\mathbf{x}' \mid k_1) = \iint_{-\infty}^{\infty} \mathscr{M}^*_{ik}(\mathbf{x} \mid \mathbf{k})\,\mathscr{M}_{jl}(\mathbf{x}' \mid \mathbf{k})\,\Phi^{(\infty)}_{kl}(\mathbf{k})\,dk_2\,dk_3.$$

A completely analogous argument can be used to show that a statistically homogeneous but otherwise random upstream entropy field, whose spectrum

$$\Phi_s(k) \equiv \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} \overline{s_\infty(\mathbf{x}-\hat{\mathbf{\imath}}U_\infty t)\,s_\infty(\mathbf{x}-\hat{\mathbf{\imath}}U_\infty t+\mathbf{y})}\,e^{-i\mathbf{k}\cdot\mathbf{y}}\,d\mathbf{y} \tag{4.8}$$

depends only on $k = |\mathbf{k}|$ owing to the assumed homogeneity, will produce a random velocity field whose correlation and one-dimensional spectrum [(4.3) and (4.7) respectively] are given by

$$R_{ij}(\mathbf{x},\mathbf{x}',\tau) = \frac{1}{(2c_p)^2}\iiint_{-\infty}^{\infty} \mathscr{N}_i^*(\mathbf{x} \mid \mathbf{k})\,\mathscr{N}_j(\mathbf{x}' \mid \mathbf{k})\,\Phi_s(k)\exp(-ik_1 U_\infty \tau)\,d\mathbf{k} \tag{4.9}$$

and

$$\Theta_{ij}(\mathbf{x},\mathbf{x}' \mid k) = \frac{1}{(2c_p)^2}\iint_{-\infty}^{\infty} \mathscr{N}_i^*(\mathbf{x} \mid \mathbf{k})\,\mathscr{N}_j(\mathbf{x}' \mid \mathbf{k})\,\Phi_s(k)\,dk_2\,dk_3. \tag{4.10}$$

5. Spectrum and correlation of turbulence produced by random entropy fluctuations

We can calculate R_{ij} and Θ_{ij} in the high frequency limit $ka \gg 1$ by substituting (3.12) into (4.2) and inserting the result together with a suitable expression for Φ_s into (4.9) and (4.10). However, it is generally not possible to express the resulting integrals in terms of elementary functions. On the other hand, this can easily be done for the case of a symmetric contraction or expansion (or combination of the two) and we shall therefore restrict our attention to this case. This type of flow occurs along the centre-line of a nozzle which can be convergent, divergent or convergent–divergent.

We shall restrict our attention to the case where the flow is continuously accelerated. Then the linearized equations should be valid even in the transonic region and, since we are dealing with high frequencies, the acoustic motion is, as explained in § 2, negligible to the order of approximation of the analysis.

Ribner & Tucker (1953) and Batchelor & Proudman (1954) both considered flows of this type. Batchelor & Proudman treat only incompressible flows while Ribner & Tucker include compressibility effects.

Since the mean flow is assumed to be symmetric in both cross-flow directions

$$U_i = \frac{\partial \Phi}{\partial x_i} = \delta_{1i} U_+ = \delta_{1i} c U_\infty, \tag{5.1}$$

$$\frac{\partial X_i}{\partial x_j} = 0 \quad \text{for} \quad i \neq j, \tag{5.2}$$

$$\frac{\partial X_2}{\partial x_2} = \frac{\partial X_3}{\partial x_3} = (\sigma c)^{\frac{1}{2}}, \tag{5.3}$$

and it therefore follows from (2.6) that

$$\partial X_1/\partial x_1 = c^{-1}, \quad \text{where} \quad c \equiv U_+/U_\infty \tag{5.4), (5.5}$$

is the ratio of U_+, the mean flow velocity downstream of the contraction, to the upstream velocity U_∞. Then since ρ_+/ρ_∞ is equal to the determinant $||\partial X_i/\partial x_j||$ (Lamb 1932, p. 14 and appendix D of I)

$$\sigma = \rho_+/\rho_\infty, \tag{5.6}$$

where ρ_+ is the mean flow density downstream of the contraction and ρ_∞ is the upstream mean flow density. Hence introducing the polar co-ordinates

$$\left.\begin{aligned} k_1 &= k\cos\theta, \\ k_2 &= k\sin\theta\cos\phi, \\ k_3 &= k\sin\theta\sin\phi, \end{aligned}\right\} \tag{5.7}$$

and using (3.8) we find that

$$\chi_1 = \frac{k}{c}\cos\theta, \quad \chi_2 = (\sigma c)^{\frac{1}{2}} k\sin\theta\cos\phi \quad \text{and} \quad \chi_3 = (\sigma c)^{\frac{1}{2}} k\sin\theta\cos\phi$$

and therefore that (3.12) becomes

$$\left.\begin{aligned} N_1 &= -N_0\,\sigma c\sin\theta, \\ N_2 &= N_0(\sigma/c)^{\frac{1}{2}}\cos\theta\cos\phi, \\ N_3 &= N_0(\sigma/c)^{\frac{1}{2}}\cos\theta\sin\phi, \end{aligned}\right\} \tag{5.8}$$

where
$$N_0 \equiv \left(\frac{1}{c} - c\right) U_\infty \frac{\sin\theta}{(\cos^2\theta/c^2 + \sigma c\sin^2\theta)}.$$

Then it follows from (4.2), (4.3), (4.8) and (4.9) that

$$\overline{u_i u_j} \equiv \overline{u_i(\mathbf{x},t)\, u_j(\mathbf{x},t)} = R_{ij}(\mathbf{x},\mathbf{x},0) = 0 \quad \text{if} \quad i \neq j$$

and that

$$\overline{u_1^2} \equiv \overline{u_1 u_1} = \left(\frac{c-c^{-1}}{2\alpha^2}\right)^2 \left[2(3-\alpha^2) - \frac{3+\alpha^2}{\sigma c^3 \alpha}\ln\left(\frac{1+\alpha}{1-\alpha}\right)\right] \left(\frac{U_\infty}{2c_p}\right)^2 \overline{s_\infty^2} \tag{5.9}$$

$$\overline{u_2^2} = \overline{u_3^2} = \frac{(c-c^{-1})^2}{4(\sigma c^3 - 1)\,\alpha^2}\left[\frac{3-\alpha^2}{2\alpha}\ln\left(\frac{1+\alpha}{1-\alpha}\right) - 3\right]\left(\frac{U_\infty}{2c_p}\right)^2 \overline{s_\infty^2}, \tag{5.10}$$

where

$$\overline{s_\infty^2} \equiv \overline{s_\infty(\mathbf{x} - \hat{\mathbf{i}} U_\infty t)\, s_\infty(\mathbf{x} - \hat{\mathbf{i}} U_\infty t)} = \iiint_{-\infty}^{\infty} \Phi_s(k)\, d\mathbf{k} = 4\pi \int_0^\infty \Phi_s(k)\, k^2 dk$$

and
$$\alpha \equiv (1 - 1/\sigma c^3)^{\frac{1}{2}}. \tag{5.11}$$

It is shown on p. 235 of Hinze (1959) that a reasonable choice for Φ_s is

$$\Phi_s(k) = \frac{\overline{s_\infty^2}}{(2\pi)^2}\left(\frac{10}{3}\right)\frac{l^3 a_0^2}{[1 + (a_0 l k)^2]^{\frac{11}{6}}}, \tag{5.12}$$

where l is the integral scale of the upstream entropy fluctuations,

$$a_0 = \frac{\Gamma(\frac{1}{3})}{\sqrt{\pi}\,\Gamma(\frac{5}{6})} \approx \tfrac{4}{3}$$

and Γ denotes the gamma function. Then inserting this into (4.10) and using (4.2) and (5.8) we find upon carrying out the integrations that

$$\Theta_{ij}(k_1) \equiv \Theta_{ij}(\mathbf{x},\mathbf{x}\mid k_1) = 0 \quad \text{if} \quad i \neq j,$$

$$\frac{\Theta_{11}(k_1)}{l\overline{s_\infty^2}(U_\infty/2c_p)^2(c-c^{-1})^2} = \frac{b_1}{[1+(a_0K_1)^2]^{\frac{5}{6}}} F\left(2,\tfrac{5}{6};2+\tfrac{11}{6};\frac{1+(\alpha a_0K_1)^2}{1+(a_0K_1)^2}\right) \tag{5.13}$$

and
$$\frac{\Theta_{22}(k_1)\,\sigma c^3}{l\overline{s_\infty^2}(U_\infty/2c_p)^2(c-c^{-1})^2} = \frac{\Theta_{33}(k_1)\,\sigma c^3}{l\overline{s_\infty^2}(U_\infty/2c_p)^2(c-c^{-1})^2}$$

$$= \frac{b_2(a_0K_1)^2}{[1+(a_0K_1)^2]^{\frac{11}{6}}} F\left(2,\tfrac{11}{6};2+\tfrac{11}{6};\frac{1+(\alpha a_0K_1)^2}{1+(a_0K_1)^2}\right), \tag{5.14}$$

where
$$K_1 \equiv lk_1, \tag{5.15}$$

$$\left.\begin{aligned} b_1 &\equiv \frac{2}{\pi}\left(\tfrac{6}{11}\right)\left(\tfrac{6}{17}\right),\\ b_2 &= \tfrac{5}{24}b_1 \end{aligned}\right\} \tag{5.16}$$

and F denotes the hypergeometric function in the usual notation.

6. Discussion of results

We shall limit the discussion to the case where the flow is continuously accelerated. Then when the mean flow is nearly incompressible (i.e. when its Mach number is not too large) σ will be unity and c will represent the contraction ratio of the stream. For compressible flow c can only be interpreted as a velocity ratio. In fact the stream has to expand when the flow is accelerated at supersonic speeds.

The energies $\overline{u_1^2}$ and $\overline{u_2^2} = \overline{u_3^2}$ in the longitudinal and transverse components of the turbulence are plotted in figure 2. The second scale applies only when the mean flow is assumed to be compressible. It represents the Mach number of the downstream flow for an upstream Mach number of 0·1. The same Mach number was used by Ribner & Tucker (1953) for their corresponding calculations of the amplification of upstream turbulence.

The solid curve corresponds to effectively incompressible mean flow. The dashed curve represents a compressible flow with an initial Mach number of 0·1. The corresponding values of σ and c were calculated from the isentropic flow relations for an ideal gas by Ribner & Tucker (1953). For nearly incompressible mean flows the energy in the transverse velocities first increases with c and then decreases while that in the axial velocities increases monotonically. However, the energy in the transverse components always remains considerably less than that in the axial component. The figure also shows that the axial energy is almost uninfluenced by compressibility until the free stream Mach number reaches a value of nearly 10! At this point the compressibility effects cause a marked reduction in the energy. The transverse velocities are, on the other hand, much more strongly influenced by compressibility which now acts to increase the energy in these components. The effect is seen to be significant even at subsonic Mach numbers.

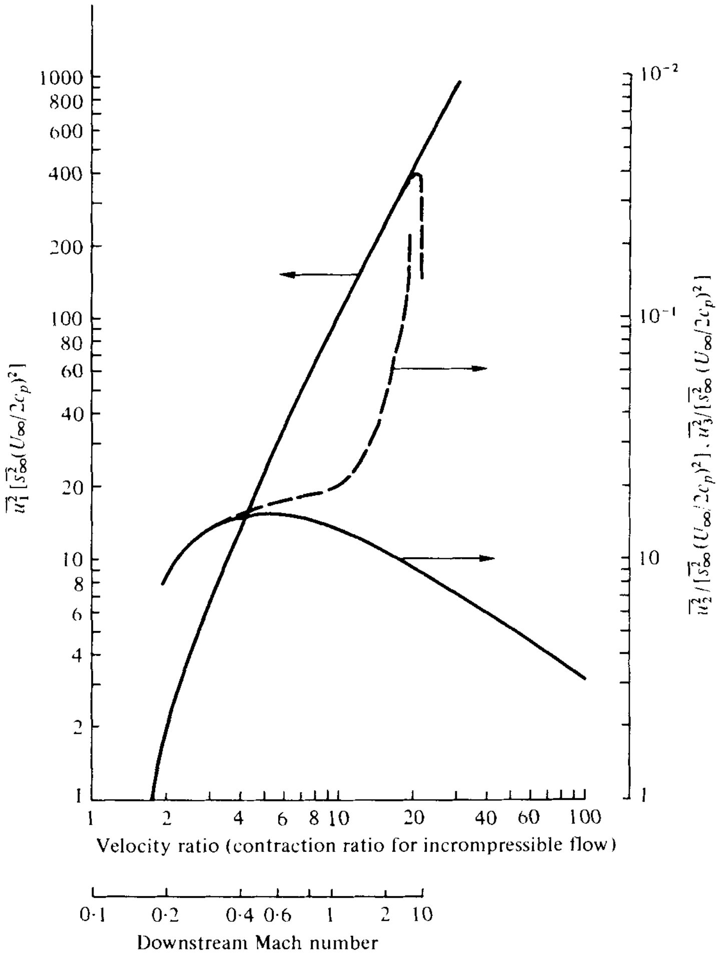

FIGURE 2. Components of turbulent energy owing to passage of entropy fluctuations through axisymmetric contractions. ——, incompressible; – – –, compressible, initial Mach number = 0·1.

In many cases of interest the contraction or velocity ratio c can be fairly large. Then since (5.11) shows that $\alpha \to 1$ as $c \to \infty$, it follows from (5.9) and (5.10) that

$$\frac{\overline{u_1^2}}{\overline{s_\infty^2}(U_\infty/2c_p)^2} \to c^2 \quad \text{as} \quad c \to \infty \tag{6.1}$$

and

$$\frac{\overline{u_2^2}}{\overline{s_\infty^2}(U_\infty/2c_p)^2} = \frac{\overline{u_3^2}}{\overline{s_\infty^2}(U_\infty/2c_p)^2} \to \frac{\ln(4\sigma c^3) - 3}{4\sigma c} \quad \text{as} \quad c \to \infty. \tag{6.2}$$

An initial random entropy field will frequently be accompanied by a corresponding turbulence field. It is therefore worthwhile comparing the preceding results with the corresponding calculations of the turbulence intensities resulting from the contraction of an initial homogeneous isotropic turbulence field. Such calculations were carried out by Batchelor & Proudman (1954) and Ribner & Tucker (1953). In fact it is shown on p. 74 of Batchelor (1953) that

$$\left.\begin{aligned} \frac{\overline{u_1^2}}{\overline{u_{\infty 1}^2}} &\to \frac{3}{4}\frac{\ln(4c^3)-1}{c^2} \\ \frac{\overline{u_2^2}}{\overline{u_{\infty 2}^2}} = \frac{\overline{u_3^2}}{\overline{u_{\infty 3}^2}} &\to \tfrac{3}{4}c \end{aligned}\right\} \quad \text{as} \quad c \to \infty \quad \text{for} \quad \sigma = 1. \tag{6.3}$$

Thus, while the energy of the initial turbulence increases with c it follows from (6.1) that it does not increase as fast as the energy in the entropy-generated turbulence. Then even if the initial entropy fluctuations are weak relative to the initial turbulence intensities, i.e. even if

$$(\overline{s_\infty^2}/2c_p)^{\frac{1}{2}} \ll (\overline{u_\infty^2}/U_\infty^2)^{\frac{1}{2}}, \tag{6.4}$$

the entropy-generated turbulence can eventually dominate that which is present in the incident stream. Since the pressure fluctuations are zero at upstream infinity, $(\overline{s_\infty^2}/c_p)^{\frac{1}{2}}$ is equal to $\beta \equiv \overline{t_\infty^2}^{\frac{1}{2}}/T_\infty$, where $\overline{t_\infty^2}^{\frac{1}{2}}$ denotes the root mean square upstream temperature fluctuation and T_∞ the mean upstream absolute temperature. Typical values of the r.m.s. temperature ratio $\beta = \overline{t_\infty^2}^{\frac{1}{2}}/T_\infty$ just downstream of a jet engine combustor are of the order of 0·03–0·07 (Pickett 1974). It is unlikely that the turbulence intensity $\overline{u_\infty^2}^{\frac{1}{2}}/U_\infty$ will exceed 0·3 in this region. It is therefore possible for the entropy-generated turbulence to exceed that initially in the stream whenever c exceeds 5. Contractions of this magnitude are not uncommon in modern turbine nozzles.

But even when this is not the case it is the energy in the axial component of entropy-generated turbulence that increases while the energy in this component decreases for the turbulence that originates upstream. Hence, the upstream entropy fluctuations can cause significant alterations in the structure of the downstream turbulence.

We are frequently interested in the turbulence energy relative to that of the local stream. Then the energies calculated above must be divided by c^2. The preceding results therefore show that the turbulent energy produced by the initial turbulence will eventually decrease relative to that of the stream while the entropy-generated turbulent energy will remain constant.

Typical one-dimensional entropy-generated turbulence spectra are shown in figure 3. The curves indicate that the axial velocity spectra decrease with increasing frequency while the transverse velocity spectra exhibit a peak near $lk_1 = 0{\cdot}6$. Results are shown for two different values of the contraction parameter $c(\sigma^{\frac{1}{3}})$. The first of these corresponds to a moderate contraction while the second corresponds to a large contraction.

For large $(\sigma^{\frac{1}{3}})\,c, \alpha \to 1$ and the hypergeometric function in (5.13) is equal to

$$|\Gamma(2+\tfrac{11}{6})\,\Gamma(1)|/|\Gamma(\tfrac{11}{6})\,\Gamma(3)| = \tfrac{1}{2}(\tfrac{17}{6})(\tfrac{11}{6}).$$

Hence, it follows from (5.13) that

$$\frac{\Theta_{11}}{\overline{s_\infty^2}(U_\infty c/c_p)^2} \to \frac{4l}{\pi[1+(aK_1)^2]^{\frac{5}{6}}} \quad \text{as} \quad c \to \infty.$$

But it is shown by Hinze (1959, p. 234) that the quantity on the right is precisely the one-dimensional spectrum corresponding to the initial spectrum (5.12) of the upstream

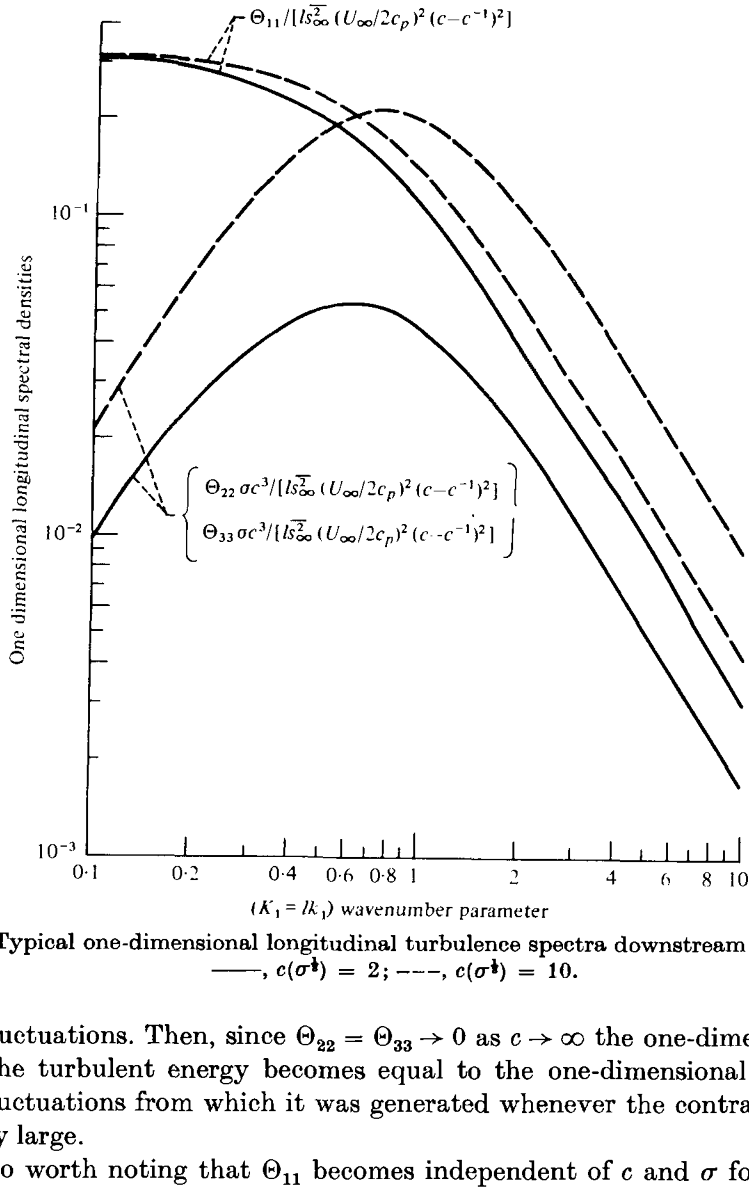

FIGURE 3. Typical one-dimensional longitudinal turbulence spectra downstream of contraction. ——, $c(\sigma^{\frac{1}{2}}) = 2$; ---, $c(\sigma^{\frac{1}{2}}) = 10$.

entropy fluctuations. Then, since $\Theta_{22} = \Theta_{33} \to 0$ as $c \to \infty$ the one-dimensional spectrum of the turbulent energy becomes equal to the one-dimensional spectrum of entropy fluctuations from which it was generated whenever the contraction ratio is sufficiently large.

It is also worth noting that Θ_{11} becomes independent of c and σ for large $c(\sigma^{\frac{1}{2}})$. The figure shows that this limit is approached quite rapidly. On the other hand the hypergeometric function in (5.14) becomes logarithmically infinite as $c(\sigma^{\frac{1}{2}}) \to \infty$ and the spectral shape always depends on this parameter, though in a very simple way.

7. Concluding remarks

We have analysed the turbulence generated by random entropy fluctuations in an accelerating stream. It is shown that the energy of the entropy-generated turbulence increases more rapidly with the contraction ratio of a subsonic flow than that of any imposed upstream turbulence. This result indicates that the entropy-generated turbulence may be more significant than the hydrodynamically generated turbulence in the turbine stages of aircraft engines.

The author would like to thank Dr Theodore Fessler for carrying out the numerical computations. After this paper was accepted for publication I became aware of a related work by Corrsin (1952), which deals with *steady* non-uniformities in the entropy of a one-dimensional flow.

Appendix

Let T_{jl} denote the tensor $\partial X_j/\partial x_l$ and let S_{ij} be its inverse, i.e. let $S_{ij}T_{jl} = \delta_{il}$. Then it follows from the Laplace development of the determinant

$$\Delta \equiv ||\partial X_i/\partial x_j||$$

that

$$\epsilon_{mnj}\Delta = \epsilon_{qpl}T_{mq}T_{np}T_{jl}$$

and since (2.15) implies that

$$\chi_i = T_{ji}k_j, \quad k_n\epsilon_{mnj}S_{km}\Delta = \epsilon_{kpl}\chi_p T_{jl}.$$

Substituting this into (3.11*b*) we obtain

$$M_{ij} = \frac{k_n\chi_r\epsilon_{kri}\epsilon_{mnj}S_{km}}{\chi^2\Delta}.$$

For incompressible flow $\Delta = 1$ (see Lamb 1932, p. 14, and appendix D of I) and this result then coincides with Batchelor & Proudman's (1954) equation (3.6).

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